Bounded walks and Catalan numbers

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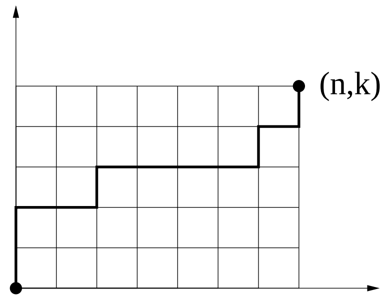
We look at walks of $n+k$ unit distance steps into North and East direction starting at the origin $(0,0)$ and ending at (n,k). The number of such paths without further restrictions is

$$\binom{n+k}{k}$$

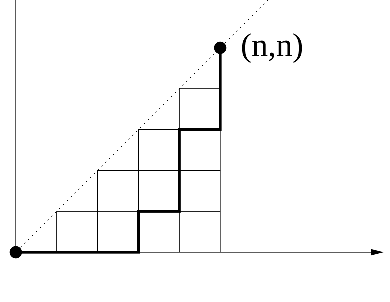
as exactly k of the $n+k$ steps are eastward steps.



This number can also be interpreted as the number of walks of $n+k$ unit distance steps on the real axis that start at 0 and end at $n-k$.

Now we consider the case where $n=k$ and look only at those paths that remain under or on the diagonal. The number of such paths is the n-th *Catalan number*

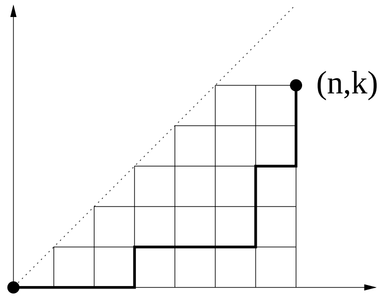
$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$



We will give a proof for this in the following. This number can also be interpreted as the number of walks of $2n$ unit distance steps on the *positive* real axis that start and end at 0.

Now we come to our main object of interest. We consider north-east paths starting at $(0, 0)$, ending at (n, k) with $n \geq k$ and remaining under the diagonal. In analogy to the binomial coefficients we will denote the number of such paths by

$$\begin{vmatrix} n+k \\ k \end{vmatrix}$$



This is also the number of walks of $n + k$ unit distance steps on the positive real axis that start at 0 and end at $n - k$.

Theorem:

$$\begin{vmatrix} n+k \\ k \end{vmatrix} = \binom{n+k}{k} - \binom{n+k}{k-1} = \frac{n-k+1}{n+1}\binom{n+k}{k}$$

The numbers $\begin{vmatrix} n+k \\ k \end{vmatrix}$ can be calculated via a "Pascal semi-triangle":

$$\begin{array}{cccccccc}
 & & & & & & & \begin{vmatrix} 0+0 \\ 0 \end{vmatrix} \\
 & & & & & & \begin{vmatrix} 1+0 \\ 0 \end{vmatrix} & \\
 & & & & & \begin{vmatrix} 2+0 \\ 0 \end{vmatrix} & & \begin{vmatrix} 1+1 \\ 1 \end{vmatrix} \\
 & & & & \begin{vmatrix} 3+0 \\ 0 \end{vmatrix} & & \begin{vmatrix} 2+1 \\ 1 \end{vmatrix} & \\
 & & & \begin{vmatrix} 4+0 \\ 0 \end{vmatrix} & & \begin{vmatrix} 3+1 \\ 1 \end{vmatrix} & & \begin{vmatrix} 2+2 \\ 2 \end{vmatrix} \\
 & & \begin{vmatrix} 5+0 \\ 0 \end{vmatrix} & & \begin{vmatrix} 4+1 \\ 1 \end{vmatrix} & & \begin{vmatrix} 3+2 \\ 2 \end{vmatrix} & \\
 & \begin{vmatrix} 6+0 \\ 0 \end{vmatrix} & & \begin{vmatrix} 5+1 \\ 1 \end{vmatrix} & & \begin{vmatrix} 4+2 \\ 2 \end{vmatrix} & & \begin{vmatrix} 3+3 \\ 3 \end{vmatrix} \\
\begin{vmatrix} 7+0 \\ 0 \end{vmatrix} & & \begin{vmatrix} 6+1 \\ 1 \end{vmatrix} & & \begin{vmatrix} 5+2 \\ 2 \end{vmatrix} & & \begin{vmatrix} 4+3 \\ 3 \end{vmatrix} & \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

The numbers in the last column are the Catalan numbers.

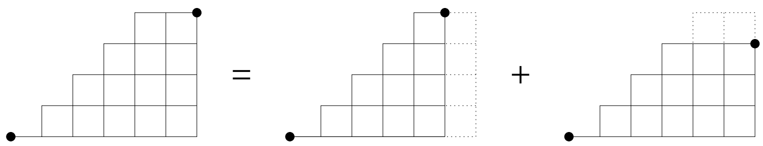
Proof:

By distinguishing between paths reaching (n,k) from the south or from the west we find the recursion formulae

$$\left|\begin{matrix} n+k \\ k \end{matrix}\right| = \left|\begin{matrix} n+k-1 \\ k \end{matrix}\right| + \left|\begin{matrix} n+k-1 \\ k-1 \end{matrix}\right|$$

for $k < n$ and $k > 0$ and

$$\left|\begin{matrix} n+n \\ n \end{matrix}\right| = \left|\begin{matrix} n+n-1 \\ n-1 \end{matrix}\right|.$$



Now we can prove the formula by induction over $n+k$. The cases where $k = 0$ are trivial (under the assumption that $\binom{a}{-1} = 0$). For $k > 0$, $k < n$ we have

$$\begin{aligned}
\left|\begin{matrix} n+k \\ k \end{matrix}\right| &= \left|\begin{matrix} n+k-1 \\ k \end{matrix}\right| + \left|\begin{matrix} n+k-1 \\ k-1 \end{matrix}\right| \\
&= \binom{n+k-1}{k} - \binom{n+k-1}{k-1} + \binom{n+k-1}{k-1} - \binom{n+k-1}{k-2} \\
&= \binom{n+k-1}{k} - \binom{n+k-1}{k-2} \\
&= \binom{n+k-1}{k} - \binom{n+k}{k-1} + \binom{n+k-1}{k-1} \\
&= \binom{n+k}{k} - \binom{n+k}{k-1}.
\end{aligned}$$

In the case where $k = n > 0$ we have

$$\begin{aligned}
\left|\begin{matrix} n+n \\ n \end{matrix}\right| &= \left|\begin{matrix} n+n-1 \\ n-1 \end{matrix}\right| \\
&= \binom{n+n-1}{n-1} - \binom{n+n-1}{n-2} \\
&= \binom{n+n-1}{n-1} - \binom{n+n}{n-1} + \binom{n+n-1}{n-1} \\
&= 2\binom{n+n-1}{n-1} - \binom{n+n}{n-1} \\
&= \binom{n+n}{n} - \binom{n+n}{n-1}.
\end{aligned}$$

This completes the proof. □

Remarks:

- If we define for $2b \le a$
$$\left|\begin{matrix} a \\ b \end{matrix}\right| := \binom{a}{b} - \binom{a}{b-1}$$
and wish to extend the definition to $b \le a$ so that
$$\left|\begin{matrix} a \\ b \end{matrix}\right| = \left|\begin{matrix} a \\ a-b \end{matrix}\right|$$
holds we have to define
$$\begin{aligned} \left|\begin{matrix} a \\ b \end{matrix}\right| \quad &:= \quad \begin{cases} \binom{a}{b} - \binom{a}{b-1} = \dfrac{a-2b+1}{a-b+1}\binom{a}{b} & \text{for} \quad 2b \le a \\[2ex] \binom{a}{b} - \binom{a}{b+1} = \dfrac{2b-a+1}{b+1}\binom{a}{b} & \text{for} \quad 2b > a \end{cases} \\ &= \quad \frac{|a-2b|+1}{\max(b, a-b)+1}\binom{a}{b}. \end{aligned}$$

- In [3] the formula
$$1 = \sum_{i=0}^{\infty} (-1)^i \binom{n-i}{i} C_{n-i}$$
is given without proof. By induction this leads to the formula
$$\left|\begin{matrix} n+k \\ k \end{matrix}\right| = \sum_{i=0}^{\infty} (-1)^i \binom{n-k-i}{i} C_{n-i}.$$

The described results are found in [1] where there are also many useful references. In [2] some of the ideas are continued. Several columns of the Pascal semi-triangle can be found in [4].

References

[1] Richard K. Guy, *Catwalks, Sandsteps and Pascal Pyramids*, Jornal of Integer Sequences, Vol. 3 (2000), Article 00.1.6.

[2] Richard K. Guy, C. Krattenthaler, Bruce E. Sagan, *Lattice paths, reflections, & dimension changing bijections*, Ars Combinatorica 34 (1992), pp. 3-15.

[3] Frank Schulz, *Sequentielles Sampling als Mittel zur Beschleunigung von Data Mining-Algorithmen*, diploma thesis, Mainz (2000).

[4] Neil J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/index.html`.