On conjecture no. 76 arising from the OEIS

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In [2] certain conjectures arising from the numerical data in the online encyclopedia of integer sequences ([3]) are presented. Problem no. 76 is to prove that the number a_n of edges in the 9-partite Turan graph of order n can be computed as the coefficient at x^n of the function

$$\frac{x}{(1-x)^2}\left(\frac{1}{1-x}-\frac{1}{1-x^9}\right).$$

The sequence (a_n) can be found in [3] as *A*033441.
Let n be a natural number and write $n = 9k + a$ with $a \in \{0, \ldots, 8\}$. Then we have

$$\begin{aligned} a_n &= \frac{1}{2}\left(a(k+1)\left((9-a)k+(a-1)(k+1)\right)+(9-a)k\left(a(k+1)+(8-a)k\right)\right) \\ &= 36k^2+8ak+\frac{a(a-1)}{2}. \end{aligned}$$

We remember the formulas

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}, \qquad \sum_{k=0}^{\infty} k^2x^k = \frac{x(1+x)}{(1-x)^3}.$$

Now we can compute the generating function F of the a_n:

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{a=0}^{8}\sum_{k=0}^{\infty}(36k^2+8ak+\frac{a(a-1)}{2})x^{9k+a} \\ &= \sum_{a=0}^{8} x^a\left(36\sum_{k=0}^{\infty}k^2x^{9k}+8a\sum_{k=0}^{\infty}kx^{9k}+\frac{a(a-1)}{2}\sum_{k=0}^{\infty}x^{9k}\right) \\ &= \sum_{a=0}^{8} x^a\left(36\cdot\frac{x^9(1+x^9)}{(1-x^9)^3}+8a\cdot\frac{x^9}{(1-x^9)^2}+\frac{a(a-1)}{2}\cdot\frac{1}{1-x^9}\right). \end{aligned}$$

With the help of a computer algebra system (like Singular, [1]) we easily verify that

$$\begin{aligned} F(x) &= \frac{x^2(1+x^3+x^6)^2(1+x+x^2)^2(1+x^4)(1+x^2)(1+x)}{(1-x^9)^3} \\ &= \frac{x}{(1-x)^2}\left(\frac{1}{1-x}-\frac{1}{1-x^9}\right), \end{aligned}$$

as conjectured. □

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References

[1] Greuel, G.-M., Pfister, G., Schönemann, H., SINGULAR 2.0, *A Computer Algebra System for Polynomial Computations*, Centre for Computer Algebra, University of Kaiserslautern (2001), `http://www.singular.uni-kl.de`.

[2] Stephan, R., *Prove or disprove 100 conjectures from the OEIS*, preprint (2004), math.CO/0409509.

[3] Sloane, N. J. A., *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/index.html`.