On conjecture no. 22 arising from the OEIS

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In [1] certain conjectures arising from the numerical data in the online encyclopedia of integer sequences ([2]) are presented. Problem no. 22 is to prove

$$n = 5^i 11^j \quad \Longrightarrow \quad n \mid \sum_{k=1}^{10} k^n.$$

We will prove a more general result:

1 Theorem
Let p, $q = 2p+1$ be odd prime numbers and i, $j \in \mathbb{N}$. Then

$$n = p^i q^j \quad \Longrightarrow \quad n \mid \sum_{k=1}^{2p} k^n.$$

Proof:
We rewrite the sum in two ways (note that n is odd):

$$\begin{aligned}\sum_{k=1}^{2p} k^n &= p^n + (2p)^n + \sum_{k=1}^{p-1} (k^n + (2p-k)^n) \\ &= p^n + (2p)^n + \sum_{k=1}^{p-1} \sum_{\mu=0}^{n-1} \binom{n}{\mu} (-k)^\mu 2^{n-\mu} p^{n-\mu}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{2p} k^n &= \sum_{k=1}^{p} (k^n + (q-k)^n) \\ &= \sum_{k=1}^{p} \sum_{\mu=0}^{n-1} \binom{n}{\mu} (-k)^\mu q^{n-\mu}.\end{aligned}$$

So it suffices to show that if p is a prime, $\alpha \in \mathbb{N}$ with $p \nmid \alpha$, $i \in \mathbb{N}$ and $\mu \in \{0, \ldots, \alpha \cdot p^i - 1\}$ then

$$p^i \mid \binom{\alpha \cdot p^i}{\mu} \cdot p^{\alpha \cdot p^i - \mu}.$$

This is easy for $\mu = 0$, so we assume

$$\mu = p^r \cdot \beta$$

with $p \nmid \beta$. Let also

$$\binom{\alpha \cdot p^i}{\mu} = p^s \cdot \gamma$$

with $p \nmid \gamma$. By a corollary of Kummer's theorem (cf. [3]) we have

$$s = \# \left\{ t \geq 0 \,:\, \mathrm{frac}\left(\frac{\mu}{p^t}\right) > \mathrm{frac}\left(\frac{\alpha \cdot p^i}{p^t}\right) \right\}$$

where frac(x) denotes the fractional part of x. Since

$$\mathrm{frac}\left(\frac{\alpha \cdot p^i}{p^t}\right) = 0$$

for $t \leq i$, we have

$$s \geq \max\{0, i - r\}.$$

We have to show that

$$s + \alpha \cdot p^i - \mu \geq i.$$

We consider two cases:

1. $r \leq i$: We have $\alpha \cdot p^i - \mu = \alpha \cdot p^i - \beta \cdot p^r \geq p^r$ and so
$$s + \alpha \cdot p^i - \mu \geq i - r + p^r > i.$$

2. $r > i$: We have $\alpha \cdot p^i - \mu = \alpha \cdot p^i - \beta \cdot p^r \geq p^i$ and so
$$s + \alpha \cdot p^i - \mu \geq \alpha \cdot p^i - \mu \geq p^i > i.$$

□

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References

[1] Stephan, R., *Prove or disprove 100 conjectures from the OEIS*, preprint (2004), math.CO/0409509.

[2] Sloane, N. J. A., *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/index.html`.

[3] Weisstein, E. W., *"Binomial Coefficient" From MathWorld–A Wolfram Web Resource*, `http://mathworld.wolfram.com/BinomialCoefficient.html`.