On conjecture no. 75 arising from the OEIS

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In [1] certain conjectures arising from the numerical data in the online encyclopedia of integer sequences ([2]) are presented. Problem no. 75 is to prove that the number a_n of (fixed) necklaces of n beads of 2 colors, 6 of them black, can be computed as the coefficient at x^n of the function

$$\frac{x^6(1-x+x^2+4x^3+2x^4+3x^6+x^7+x^8)}{(1-x)^4(1+x)^2(1-x^3)(1-x^6)}.$$

The sequence (a_n) can be found in [2] as $A032191$.
Let A_n be the set of strings of n beads of 2 colors, 6 of them black. The cyclic group G_n of order n operates on A_n by rotation, and the number a_n is the number of orbits in A_n under the action of G_n.
If $n \equiv 1,5 \mod 6$ then the stabilizer of an element $a \in A_n$ must be trivial and we have $a_n = \frac{1}{n} \cdot \binom{n}{6}$.
If $n \equiv 2 \mod 6$ then the stabilizer of an element $a \in A_n$ has order 1 or 2. If it has order 2 then a must be invariant under rotation by $n/2$. There are exactly $\binom{n/2}{3}$ such elements. We find

$$a_n = \frac{\binom{n}{6} - \binom{n/2}{3}}{n} + \frac{2 \cdot \binom{n/2}{3}}{n} = \frac{1}{n} \cdot (\binom{n}{6} + \binom{n/2}{3}).$$

If $n \equiv 3 \mod 6$ then the stabilizer of an element $a \in A_n$ has order 1 or 3. If it has order 3 then a must be invariant under rotation by $n/3$. There are exactly $\binom{n/3}{2}$ such elements. We find

$$a_n = \frac{\binom{n}{6} - \binom{n/3}{2}}{n} + \frac{3 \cdot \binom{n/3}{2}}{n} = \frac{1}{n} \cdot (\binom{n}{6} + 2\binom{n/3}{2}).$$

If $n \equiv 3 \mod 6$ then the stabilizer of an element $a \in A_n$ has order 1, 2, 3 or 6. The last three possibilities correspond to invariance under rotation by $n/2$, $n/3$ resp. $n/6$. We find

$$\begin{aligned} a_n &= \frac{\binom{n}{6} - \binom{n/2}{3} - \binom{n/3}{2} + \binom{n/6}{1}}{n} + \frac{2(\binom{n/2}{3} - \binom{n/6}{1})}{n} + \frac{3(\binom{n/3}{2} - \binom{n/6}{1})}{n} + \frac{6\binom{n/6}{1}}{n} \\ &= \frac{1}{n} \cdot (\binom{n}{6} + \binom{n/2}{3} + 2\binom{n/3}{2} + \frac{n}{3}). \end{aligned}$$

Altogether we have the formula

$$a_n = \begin{cases} 0, & n < 6, \\ \frac{1}{n} \cdot \binom{n}{6}, & n \equiv 1, 5 \mod 6, \\ \frac{1}{n} \cdot (\binom{n}{6} + \binom{n/2}{3}), & n \equiv 2, 4 \mod 6, \\ \frac{1}{n} \cdot (\binom{n}{6} + 2\binom{n/3}{2}), & n \equiv 3 \mod 6, \\ \frac{1}{n} \cdot (\binom{n}{6} + \binom{n/2}{3} + 2\binom{n/3}{2} + \frac{n}{3}), & n \equiv 0 \mod 6. \end{cases}$$

Now the generating function of the a_n is

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=6}^{\infty} \frac{\binom{n}{6}}{n} x^n + \sum_{n=0}^{\infty} \frac{\binom{n+3}{3}}{2n+6} x^{2n+6} + 2 \sum_{n=0}^{\infty} \frac{\binom{n+2}{2}}{3n+6} x^{3n+6} + \frac{1}{3} \sum_{n=0}^{\infty} x^{6n+6} \\ &= x^6 \cdot \left(\sum_{n=0}^{\infty} \frac{\binom{n+6}{6}}{n+6} x^n + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\binom{n+3}{3}}{n+3} x^{2n} + \frac{2}{3} \sum_{n=0}^{\infty} \frac{\binom{n+2}{2}}{n+2} x^{3n} + \frac{1}{3} \sum_{n=0}^{\infty} x^{6n} \right). \end{aligned}$$

Using the formula

$$\sum_{n=0}^{\infty} \frac{\binom{n+k}{k}}{n+k} x^n = \frac{1}{k} \cdot \frac{1}{(1-x)^k}$$

we conclude that

$$\begin{aligned} F(x) &= x^6 \cdot \left(\sum_{n=0}^{\infty} \frac{\binom{n+6}{6}}{n+6} x^n + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\binom{n+3}{3}}{n+3} x^{2n} + \frac{2}{3} \sum_{n=0}^{\infty} \frac{\binom{n+2}{2}}{n+2} x^{3n} + \frac{1}{3} \sum_{n=0}^{\infty} x^{6n} \right) \\ &= x^6 \cdot \left(\frac{1}{6} \cdot \frac{1}{(1-x)^6} + \frac{1}{6} \cdot \frac{1}{(1-x^2)^3} + \frac{1}{3} \cdot \frac{1}{(1-x^3)^2} + \frac{1}{3} \cdot \frac{1}{1-x^6} \right). \end{aligned}$$

Now it is easy to compute that

$$F(x) = \frac{x^6(1 - x + x^2 + 4x^3 + 2x^4 + 3x^6 + x^7 + x^8)}{(1-x)^4(1+x)^2(1-x^3)(1-x^6)},$$

as conjectured. □

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References

[1] Stephan, R., *Prove or disprove 100 conjectures from the OEIS*, preprint (2004), math.CO/0409509.

[2] Sloane, N. J. A., *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/index.html`.