Modularity of Calabi–Yau threefolds

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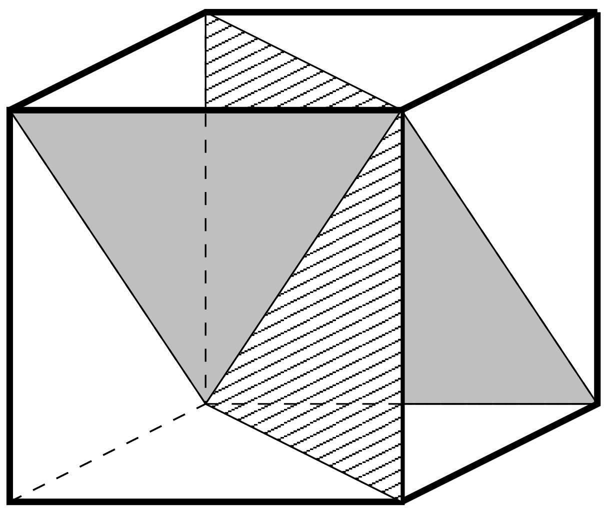


Calabi–Yau d-folds

- X smooth projective variety of dimension d defined over $\mathbb{Z}$
- X Calabi–Yau $:\Longleftrightarrow$ $K_X \simeq O_X$, $H^i(X, \mathcal{O}_X) = 0$, $0 < i < d$
- $d = 1$: elliptic curve
- $d = 2$: K3 surface
- $d = 3$: Calabi–Yau threefold

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(X) & & 0 & \\
1 & & h^{2,1}(X) & & h^{1,2}(X) & & 1 \\
 & 0 & & h^{2,2}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

$$
\begin{array}{rl}
b^0(X) & = 1 \\
b^1(X) & = 0 \\
b^2(X) & = h^{1,1}(X) \\
b^3(X) & = 2(1 + h^{2,1}(X)) \\
b^4(X) & = h^{2,2}(X) = h^{1,1}(X) \\
b^5(X) & = 0 \\
b^6(X) & = 1 \\
\hline
\chi(X) & = 2(h^{1,1}(X) - h^{2,1}(X))
\end{array}
$$

$$
\begin{array}{rcl}
h^{2,1}(X) & \simeq & \text{no. of complex deformations of } X \\
h^{1,1}(X) & = & \mathrm{rk}(\mathrm{Pic}(X))
\end{array}
$$

Modularity of Calabi–Yau d-folds

- $\#X_p :=$ no. of points on X over $\mathbb{F}_p$
- Modularity $\simeq$ for good primes p, $\#X_p$ is determined by coefficients of certain modular forms

- ℓ-adic cohomology, Frobenius map, Galois representations, L-series

- Lefschetz fixed point formula

$$\#X_p = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}(\mathrm{Frob}_p^* \,|\, H^i_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell))$$

- Elliptic curves:

$$\#E_p = 1 - a_p + p$$

Wiles et al.: a_p are the coefficients of a weight 2 newform for $\Gamma_0(N)$, with $N =$ Conductor of E

- Calabi–Yau threefolds:

$$\#X_p = p^3 + 1 + k_p \cdot (p^2 + p) - a_p$$

Conjecture: a_p for good p determined by modular forms

Modularity of Calabi–Yau threefolds

- X rigid, i.e., $h^{2,1}(X) = 0$. Precise conjecture: a_p are the coefficients of a weight 4 newform for $\Gamma_0(N)$, with N divisible only by bad primes

- X non–rigid, i.e., $h^{2,1}(X) > 0$. Special case ("splitting"):

$$a_p = b_p + p \cdot (*)$$

- Both cases are accessible to computer searches

- Coefficients of modular forms: HECKE by W. Stein

Recipe

1. Start with singular Calabi–Yau threefold X defined over $\mathbb{Z}$
2. Study singularities of X, determine bad primes
3. Resolve singularities, obtain smooth model $\tilde{X}$ of X
4. Determine Hodge numbers
5. Determine $\#\tilde{X}_p$ by counting points with a computer
6. Compare a_p with coefficients of modular forms
7. Prove modularity (f.i. Serre–Livné–Faltings)

Construction of Calabi–Yau threefolds

- Quintic in $\mathbb{P}^4$
- Intersection of two cubics in $\mathbb{P}^5$
- Intersection of a quartic and a quadric in $\mathbb{P}^5$
- Intersection of a cubic and two quadrics in $\mathbb{P}^6$
- Intersection of four quadrics in $\mathbb{P}^7$
- Complete intersections in weighted projective spaces
- Toric constructions
- Fibre products of elliptic fibrations
- Triple sextic = triple covering of $\mathbb{P}^3$
  branched along sextic surface
- Double octic = double covering of $\mathbb{P}^3$
  branched along octic surface

$$u^2 = f_8(x, y, z, t)$$

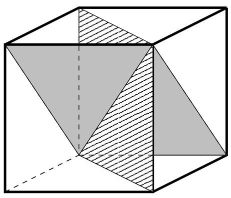
  – octic arrangements

Double octics constructed from eight planes with rational coefficients

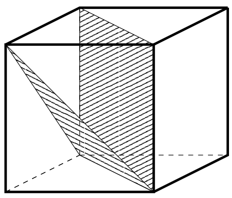
$h^{2,1}$	no. of examples
9	1
8	1
7	5
6	12
5	36
4	76
3	125
2	120
1	63
0	11
	450

- The rigid examples are modular
- There are many non–rigid examples with “splitting”

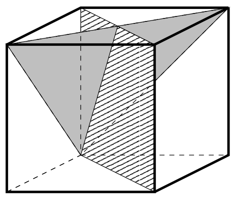
Rigid double octics constructed from eight planes



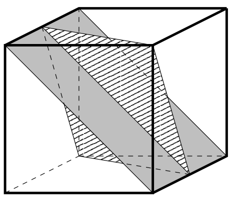
8/1, $h^{1,1} = 70$



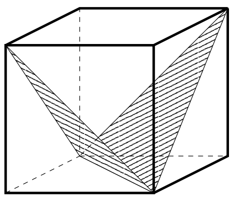
32/2, $h^{1,1} = 62$



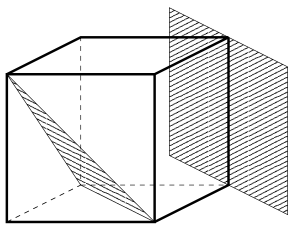
32/1, $h^{1,1} = 54$



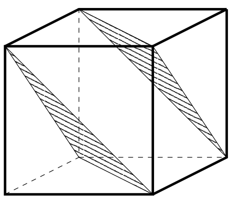
8/1, $h^{1,1} = 50$



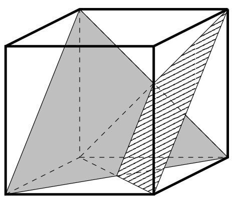
8/1, $h^{1,1} = 50$



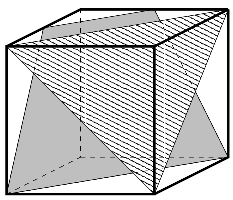
8/1, $h^{1,1} = 46$



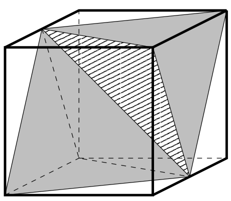
8/1, $h^{1,1} = 44$



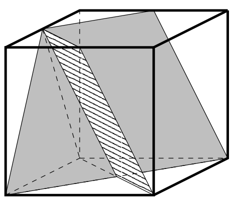
12/1, $h^{1,1} = 40$



6/1, $h^{1,1} = 40$



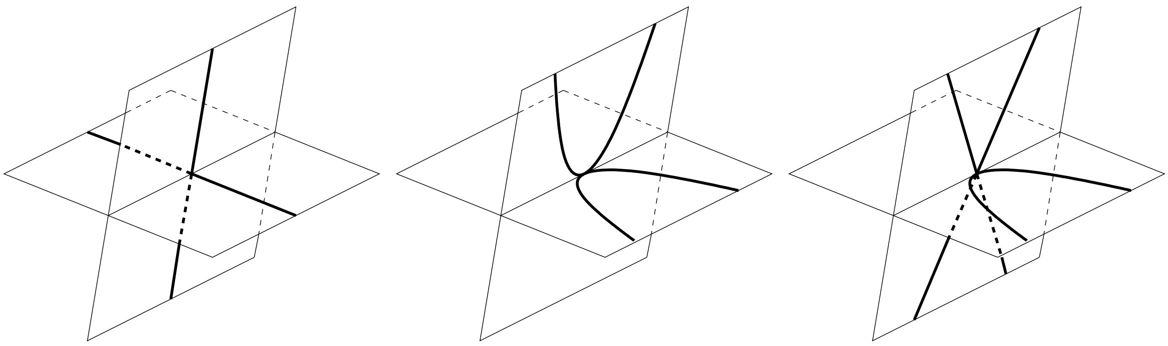
8/1, $h^{1,1} = 40$



6/1, $h^{1,1} = 38$

Six planes and a smooth quadric

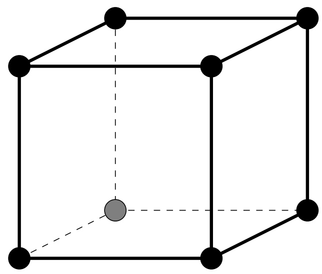
- double point
- 3 types of triple points



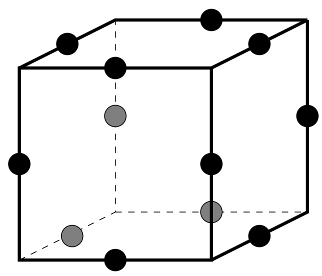
- 8 types of fourfold points
- 15 types of fivefold points

$\rightsquigarrow \qquad \geq 19258$ examples

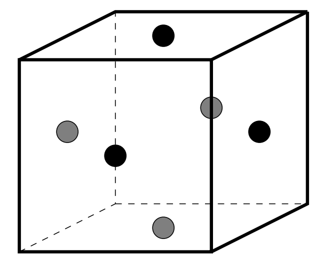
Six planes and a quadric, examples



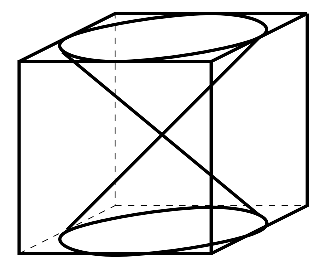
$h^{2,1} = 2$



$h^{2,1} = 0$



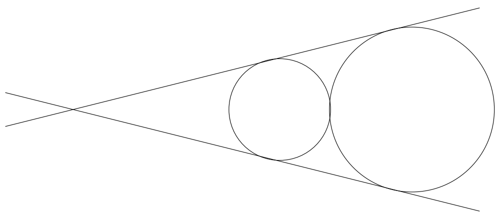
$h^{2,1} = 3$



$h^{2,1} = 1$

More double octics

- 4 planes and 2 quadrics



$h^{2,1} = 1$

- 4 quadrics
- Kummer surface and 4 planes
- cubic and 5 planes
- Clebsch cubic and S_5-symmetric quintics

$$C_1 = C_3 \cdot (aC_2C_3 - bC_5) = 0$$

$(a : b) = (5 : 12)$: Barth's quintic with 15 cusps,
bad reduction at 13

- S_5-symmetric octics

$\rightsquigarrow$ examples with bad reduction at 11, 13, 19 or 37.

- Sarti's Heisenberg-invariant surfaces

Bad primes

- Bad primes occurring in levels:

$$2, 3, 5, 7, 11, 13, 17, 19, 31, 37, 73$$

Maximum powers: 8 for $p = 2$, 5 for $p = 3$, 2 else

- Conductor = Level of an elliptic curve E:

$$N = \prod_{p \text{ prime}} p^{f_p}$$

where

$$f_p = \begin{cases} 0, & \text{if } E \text{ has good reduction modulo } p, \\ 1, & \text{if } E \text{ has a node modulo } p, \\ 2, & \text{if } E \text{ has a cusp modulo } p, \text{ and } p \neq 2, 3, \\ 2 + \delta_p, & \text{if } E \text{ has a cusp at } p = 2 \text{ or } p = 3. \end{cases}$$

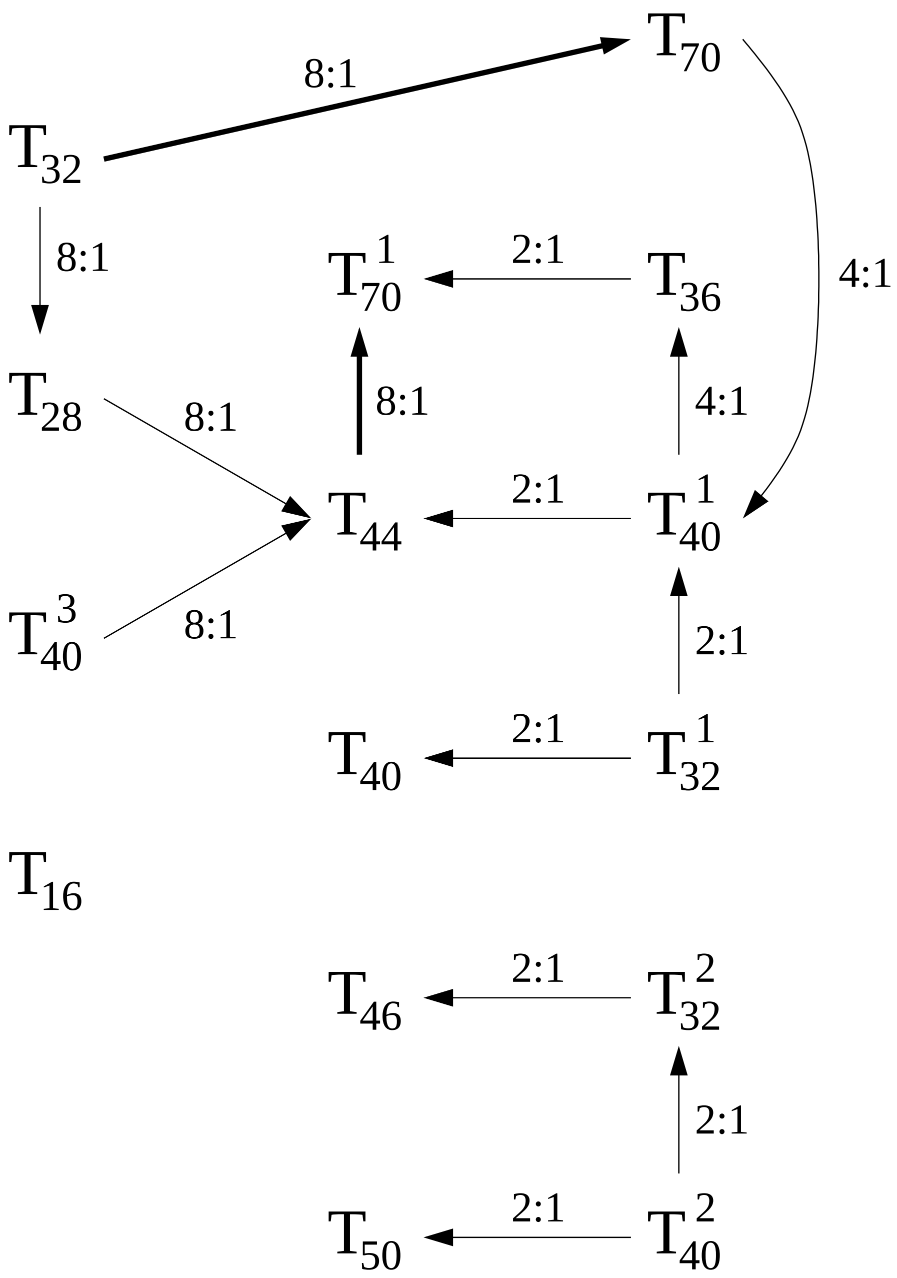
- "rule of thumb" for Calabi–Yau threefolds:

  $p^2 \mid N \rightsquigarrow$ non–isolated singularities over $\mathbb{F}_p$

  $p \mid N$, $p^2 \nmid N \rightsquigarrow$ isolated singularities over $\mathbb{F}_p$

  $p \nmid N \rightsquigarrow$ smooth over $\mathbb{F}_p$

Correspondences



Correspondences for level 8 rigid Calabi–Yau threefolds

Future projects

- Find examples with large bad primes
- Find more correspondences
- Complete modularity proof for rigid Calabi–Yau threefolds
- Study non–rigid Calabi–Yau threefolds
- E.g., classify the cases with $h^{2,1} = 1$
- Prove modularity for all Calabi–Yau threefolds
- Application to mirror symmetry and physics
- Move on to the next dimension